Complex Exponentials and RLC Circuits

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1 Introduction

Students that have taken an introductory course on classical mechanics should be familiar with the harmonic oscillator (and its relatives the damped oscillator and the forced oscillator). Afterward in electromagnetism it is revealed that current in an RLC circuit exhibits the same behavior as velocity in an oscillator. Typically this behavior is represented mathematically using the two independent trigonometric functions *sine* and *cosine*. Herein we will explore the mathematical representation that uses the more elegant complex exponential functions.

2 Complex Numbers

A complex number is a number with both a real part and an imaginary part. Any complex number can be written in the form $z = x + iy$ where x and y are real numbers. Perhaps less obvious is that we can also write any complex number in the following form:

$$z = re^{i\phi} \tag{1}$$

We can return to the x, y form of z by invoking *Euler's Equation* which follows from the series expansion of the exponential function:

$$e^{i\phi} = \cos\phi + i\sin\phi \tag{2}$$

Then,

$$x = r\cos\phi$$
$$y = r\sin\phi$$

Or, if we want to go from x, y to r, ϕ,

$$r = \sqrt{x^2 + y^2} = (x + iy)(x - iy)$$
$$\phi = \arctan\frac{y}{x}$$

Looking at these equations, there are a few things that should stick out. The first is that we were able to obtain a real, positive number by multiplying z with itself after flipping the sign of the imaginary part. This is known as *complex conjucation* and is denoted by $\overline{z}$.

The second thing that you may have noticed is that complex numbers behave just like vectors in two dimensions with the real part corresponding to the x-component and the imaginary part corresponding to the y-component. The r, ϕ representation of z, then, corresponds to the polar coordinates of z where r is the magnitude and ϕ is the phase. Of course, while the vector concept may make complex numbers easier to visualize, we should not forget that they are scalars and consequently can be manipulated with the same ease as real scalars.

3 Ordinary Differential Equations

A differential equation is an equation involving not just a function but its derivatives as well (an ordinary differential equation is just a differential equation involving only one independent variable). The methods for solving ordinary differential equations are enough to fill multiple textbooks so here I intend only to state a few important properties of differential equations.

Consider the following first order differential equation:

$$\dot{x} + x = 0$$

This equation represents exponential decay with the following solution:

$$x(t) = Ce^{-t}$$

This is perhaps the simplest non-trivial differential equation that we could have chosen. Still, it possesses some interesting properties that we ought to stop and analyze for a moment. Actually, there is really only one feature of the solution to this equation that I would like to mention and that is the C in the front of the equation. This C should be familiar to any student that has taken an introductory calculus course– it is a constant of integration. What this tells us is that the solution to our differential equation is not unique; any scalar multiple of e^{-t} will also satisfy the differential equation.

Let's look now at a second order differential equation:

$$\ddot{x} + x = 0$$

This new differential equation looks very similar to the last one but now instead of an exponential decay we get oscillation manifested in the following equation:

$$x(t) = A \sin t + B \cos t$$

Before we had only one constant of integration; now we have two. Moreover, we now have two distinct functions composing our solution. What does this mean? It means that functions are vectors! Of course, you may now be asking what it is that I mean by that. Well, just look at sine and cosine. Neither one is a scalar multiple of the other. In other words, they are *linearly independent* functions. Hence, $x(t)$ is a function in a two-dimensional *function space* spanned by the basis functions *sine* and *cosine*.

"So what?" you might be asking. "What benefit is there to thinking of functions as vectors?" There are actually quite a few benefits, but one in particular that I want to emphasize involves the notion of basis vectors. An elementary property of vector spaces is that an n-dimensional vector space can be spanned by any n linearly independent basis vectors. Similarly, we can span our two-dimensional using *any* two linearly independent functions that satisfy our differential equation. In one dimension this fact is trivial but in two dimensions we can now make linear combinations of our original basis functions in order to construct new basis functions. For instance:

$$e^{it} = \cos t + i \sin t$$

$$e^{-it} = \cos t - i \sin t$$

In this new basis we can write $x(t)$ as,

$$x(t) = Ae^{it} + Be^{-it}$$

We will see shortly why this basis is often more convenient to work in than the sine/cosine basis.

4 The Harmonic Oscillator ODE

The previous differential equation was an example of a harmonic oscillator differential equation. In electrodynamics we have a more general form of this equation for LC circuits:

$$L\ddot{q} + \frac{1}{C}q = 0$$

Let's try writing the solution to this equation in the complex exponential basis:

$$q(t) = Ae^{i\frac{t}{\sqrt{LC}}} + Be^{-i\frac{t}{\sqrt{LC}}}$$

Of course, typically we are more interested in current than charge. We can replace q with $I = \dot{q}$ in the above equation and our differential equation will still be satisfied:

$$I(t) = Ae^{i\frac{t}{\sqrt{LC}}} + Be^{-i\frac{t}{\sqrt{LC}}}$$

So far all of this should seem perfectly legitimate from a mathematical perspective– we needed an equation that solved our differential equation and the above equation just does that. But what about from a physical perspective? What do we do about those pesky is? Do they have some sort of physical interpretation? Well yes, they do. Do you remember the following formula?

$$z = re^{i\phi}$$

If we take the derivative of this function with respect to ϕ we get,

$$\frac{dz}{d\phi} = ire^{i\phi}$$

From Euler's Equation we can write i as,

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\frac{\pi}{2}}$$

In other words,

$$\frac{dz}{d\phi} = re^{i\frac{\pi}{2}}e^{i\phi} = re^{i\left(\phi+\frac{pi}{2}\right)}$$

The i corresponds to a 90° phase shift! This shouldn't be too surprising, though. In the complex plane, multiplying by i takes us from the real axis to the imaginary axis, i.e. from the x-axis to the y-axis which corresponds to a 90° rotation. Moreover, taking the derivative of one of our previous basis functions, e.g. sine gives us cosine which is just sine shifted by 90°. By taking the derivative twice we get a -1 corresponding to a 180° phase shift which just happens to be the phase shift between capacitor and inductor voltages.

All well and good, but what about the magnitude of our function? It doesn't make any sense to talk about imaginary amplitude. My suggestion here is to try to take the math less literally. For our purposes, mathematics is, after all, merely a tool for extracting the physical quantities that we are interested in. For this purpose, the fact that our function has an imaginary component is not an issue. Once we have our equation it is a trivial matter to extract the amplitude and phase which will most often be what we are itnerested in.

Still, if you are not completely satisfied with this explanation, then I offer you another. You may have noticed that we have not yet made any assumptions about initial conditions. Naturally, our initial conditions are always going to be real because physical quantities are always going to be real. If we go ahead and plug these initial conditions into our equation then we will see that all the imaginary parts go away and we are left with sine and cosine once again. This can be quite a hassle, though, and typically we don't care where our function starts (as with potential energy, what we care about when it comes to phase is the phase difference *between* two quantities, not the absolute phase). So, what we can do is leave our function as a complex exponential until such a time comes that we want to get rid of the imaginary part. What we then do is take the real part of our function. This will give us a nice sine or cosine (or anything in between) function with amplitude, frequency, and phase shift in tact. This may seem like a shady technique but this is basically what plugging in intial conditions does anyway and, as mentioned, all of the parameters that we are interested in are preserved which is ultimately all that matters.

5 The Forced Oscillator ODE

Now that all the basics of complex exponentials and their use in solving differential equations have been covered I would like to conclude with an example where the benefit of complex exponentials over trigonometric functions can be readily seen. Consider the following differential equation for a forced harmonic oscillator:

$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = \mathcal{E}\cos\omega t$$

We want to find the resonant frequency and the current amplitude. We will do this first using a sine/cosine and then again using a complex exponential solution.

Sine/Cosine Solution:
Let $I(t) = A\cos\omega t$. This will most likely not satisfy the differential equation as it was given but because we don't actually care about absolute phase we are free to phase shift the forcing term to fit our solution (in other words, the $\cos\omega t$ in the forcing term does not literally mean $\cos\omega t$ but merely that this term is oscillating with frequency ω). Then,

$$q = \int I dt = \frac{A}{\omega}\sin\omega t$$

$$\ddot{q} = \frac{dI}{dt} = -A\omega\sin\omega t$$

Plugging these into our differential equation and rearranging we get,

$$A\left(R\cos\omega t + \left(\frac{1}{\omega C} - \omega L\right)\sin\omega t\right) = \mathcal{E}\cos(\omega t + \phi)$$

Combining the sine and cosine terms will require the painful application of trigonometric identities. Since we aren't interested in the absolute phase and only in the amplitude, though, we can find the amplitude by drawing a triangle with sides x and y, hypotenuse r, and angle ωt. Then,

$$x = AR\cos\omega t$$
$$y = A\left(\frac{1}{\omega C} - \omega L\right)\sin\omega t$$
$$r = \sqrt{x^2 + y^2} = \mathcal{E}$$

Hence,

$$A = \frac{\mathcal{E}}{\sqrt{R^2 + \left(\frac{1}{\omega C} - \omega L\right)^2}}$$
$$\Rightarrow \omega_r = \frac{1}{\sqrt{LC}}$$

Fortunately that wasn't too painfully since we didn't actually have to invoke any trigonometric identities thanks to our triangle trick. Still, let's see how things go using complex exponentials.

Complex Exponential Solution:
Let $I(t) = Ae^{i\omega t}$. Then,

$$q = \frac{A}{i\omega}e^{i\omega t} = -i\frac{A}{\omega}e^{i\omega t}$$
$$\ddot{q} = iA\omega e^{i\omega t}$$

Plugging this into the differential equation we get,

$$A\left(R + i\left(L\omega - \frac{1}{\omega C}\right)\right)e^{i\omega t} = \mathcal{E}e^{i\omega t}$$

Notice how the exponential term factored right out of the equation. If we wanted to factor out the phase term in a sine/cosine expression we would have to invoke trigonometric identities. All we need to do now is take the magnitude of both sides and rearrange.

$$A\sqrt{\left(R+i\left(L\omega-\frac{1}{\omega C}\right)\right)\overline{\left(R+i\left(L\omega-\frac{1}{\omega C}\right)\right)}}=A\sqrt{R^2+\left(L\omega-\frac{1}{\omega C}\right)^2}=\mathcal{E}$$

$$A=\frac{\mathcal{E}}{\sqrt{R^2+\left(\frac{1}{\omega C}-\omega L\right)^2}}$$

$$\Rightarrow \omega_r=\frac{1}{\sqrt{LC}}$$